**COSC 326 - Etude 3 [Cubes]**

**Solution:**

Answer:

Given we have eight 1 x 1 x 1 cubes that are coloured blue or yellow, and arrange them into a 2 x 2 x 2 dimension cube in any combination, we will get 23 unique pattern arrangements.

Simplifying the problem:

To solve this problem it is better to simplify the problem and think of it as finding the unique pattern arrangements when colouring the 8 vertices of a cube. This is analogous to finding distinct cube colourings because a colour is the same for all three faces for a vertex just like a face subsection. When we don’t consider the rotations, we can easily work out the number of distinct pattern arrangements. It becomes more difficult with rotations because patterns are considered the same if they differ by rotational transformation. This is because in a cube there are multiple rotational symmetries (orbits) - which is related to the Burnside’s Lemma. For more clarity, an orbit (or fix) is when a pattern remains the same given a motion - in this case a rotational transformation. The Burnside’s lemma is useful for this problem because it is a orbit-counting formula where the number of orbits is equal the average number of transformational fixes (Proof Wiki, 2017).

Defining Burnside’s Lemma:

To find the number of orbits of a polygon we use the following equation. The Burnside’s lemma is defined under this equation:

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Notations:

* Let X be a set of fixed elements.
* Let G be a finite group acting on a set X.
* Let X/G be the set of orbits under group of actions G.
* For each g in G, let denote the set of all fixed elements by the group element g.

(Proof Wiki, 2017).

Interpreting Burnside’s Lemma:

Here are some more notations to explain our process:

* Let C be the number colours possible to colour vertices.
* Let v be the number of vertices.
* Let x = be the permutations of colour patterns given number of fixed vertices v.
* For this lemma, \* denote the transformation g applied to x.

For our pattern arrangement counting problem, we interpret the notations based on a cube. G will be a group of cube rotational transformations that is applied to a set X. Set X will be all possible fixed patterns given a motion from group G. For example given g = of a cube face rotation, and given x = , applying transformation (g \* x) will determine the number of fixed patterns by motion g. For our cube colouring problem to find all fixed patterns given by motion g from Set G, we first need to determine G which is the group of rotational transformations which is logically implied from counting cube symmetries.

Symmetries of a Cube (Defining Group G):

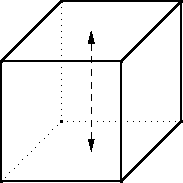
We can compute the number of group members of G for any equilateral 3D shape by multiplying the number of vertices and the number edges for one vertex:

(Nal, 2017).

This gives use 24 elements in group G. We can confirm this by finding all axis of symmetries of a cube.

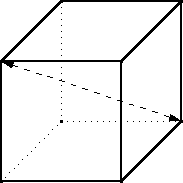
The first cube symmetry is when the cube is has no motion. For any group in Burnside’s lemma there will always be an identity where the group has no motion. So, if a cube has no motion and we can colour the vertices with 2 colours, then we get a total of (256) distinct fixed patterns - which becomes the identity fixture of this problem. This becomes the first motion in our group G.

The second cube axes of symmetries is when the axis goes through the faces of the cube. We can have, , rotations along each of this symmetry. We can also classify this rotation by saying , because -is the same as . So for one axes we get 3 motions (3 group members). For face rotations we have 3 pairs of faces so in total we have 9 group members for face rotations.

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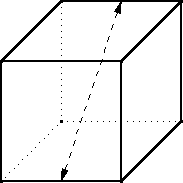
Demonstrates rotation around face axis (Dominus, 2006).

The third cube axes of symmetry is when the axis goes through opposite vertices. We can have , rotation along each of this symmetry. We can also classify this rotation by saying , rotations because is the same as , and therefore we get 2 motions per axis (2 group members). For vertex rotations we have 4 pairs of vertices so in total we have 8 group members for vertex rotations.

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Demonstrates rotation around vertex axis (Dominus, 2006).

The final cube axes of symmetry is when the axis goes through opposite edges. We have a rotations for each edge axes. For edge rotations we have one motion and 6 different pairs of edges. So in total we have 6 motion groups for edge rotations.



Demonstrates rotation around edge axis (Dominus, 2006).

If we add all the motion group elements:

* 1 for identity.
* 9 for face rotations.
* 8 for vertex rotations.
* 6 for edge rotations.

We get a total of 24 group members: 1 + 9 + 8 + 6 = 24.

Counting Number of Orbits:

|  |  |
| --- | --- |
| **Rotation Type** | **Cycle Structure** |
| Identity (0 Degrees) | 1, 1, 1, 1, 1, 1, 1, 1 |
| Face Rotation (9) | 2, 2, 2, 2 |
| Vertex Rotation (8) | 1, 1, 3, 3 |
| Edge Rotation (6) | 4, 4 |

Table shows the number of vertices for each orbit given rotation from group G.

We already know for the identity motion from group G we get 256 fixed patterns because with no motion, each vertices are in its own orbit (or cycle) - in other words, each vertex stays in its own position. In general given C is the number of colours we get .

For face rotations there will always be a pair of vertices in the same orbit. For example if you look at a face along the axis of symmetry you will see a square (front and back face). For each front vertex are parallel with the back vertices. When applying amotion, the pair of vertices in parallel will always be the same forrotations because those pairs will be shifted by. Formotion, those pair of vertices will appear on the opposite diagonal for the square, and therefore there are always 2 vertices rotating in its own orbit. This is how we get the cycle structure (2, 2, 2, 2) shown in the table. So in total there are 4 different orbits. In general given C is the number of colours we get .

For vertex rotations there will always be 2 vertices each in their own orbit and 2 groups of 3 vertices in different orbits. For example if we look at an axis along the diagonal pairs, the 2 vertices along the axis will be in its own orbit because they are stationary. There are 3 vertices that connects via edges to each vertex along the axis which will be in their own orbit. There will be the same set of vertices that will orbit in this same structure if we looked at the vertex along the axis from the opposite side. So therefore, we get the cycle structure (1, 1, 3, 3) shown in the table. In total there will be 4 separate orbits. In general given C is the number of colours we get .

For edge rotations there will be 2 groups of 4 vertices in different orbits. For example if we look along the axis from one edge, we will see two faces one each side of the axis. For amotion those faces will swap to the opposite side of each other - which means the 4 vertices that make up those faces will have their own orbit. So we get a cycle structure (4, 4) shown in the table. In total there will be 2 separate orbits. In general given C is the number of colours we get .

Given the above logic we have a general formula for counting the vertices of a cube:

Now if C = 2 (Yellow and Blue):

We get:

Therefore we get 23 unique patterns of a cube using 2 colours (yellow and blue).

**How can you ensure you have not counted any arrangement twice and have found them all?**

The Burnside’s lemma counts the number of different orbits of a colour set where a rotational group acts on, and has been proven to work out the number of different orbits.

There are 3 properties to a cube: faces, vertices, and edges. Because the cube is a uniform shape, we can only use faces, vertices and edges to divide the cube symmetrically - shown with the above logic. We can also confirm the group members by computing: . So we can ensure that we have found the proper amount of g elements in group G.

We have the initial rotation of . So a identity rotational symmetry of 1.

When we rotate by faces there are 4 vertices and because it is a square, the minimum rotation is (we can only rotate by factors of ). So that is why we can only rotate is 3 possible ways (, , ), and because we have 3 pairs of faces, there are a total of 9 symmetrical rotations.

When we rotate by vertices there are 8 vertices. If we look along the axis of one side, there are 3 subsections to the cube (we see 3 faces that creates a vertex). So the minimum rotation is (we can only rotate by factors of ) and that is why we have 2 rotations for each vertex pair. Since there are 4 pairs of vertices, we get a total of 8 symmetrical rotations.

When we rotate by edges there the minimum rotation is . If we look along the axis of diagonally complementary edges, there will be two faces one each side. To complete a valid rotation the minimum angle is . Because there are 6 pair of diagonally complementary edges, we get a total of 6 symmetrical rotations.

With this logic, we have the correct amount motions in group G. Since we have the right amount of motions, for each motion we can methodically derive the correct number of fixes with no duplicates (shown in the solution section).

**References:**

Dominus, M. (2006). *Counting Squares.* [Online] Available from: https://blog.plover.com/math/polya-burnside.html [Accessed 13th January 2018].

Proof Wiki. (2017). *Burnside’s Lemma.* [Online] Available from: https://proofwiki.org/wiki/Burnside%27s\_Lemma [Accessed 21 January 2018].

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